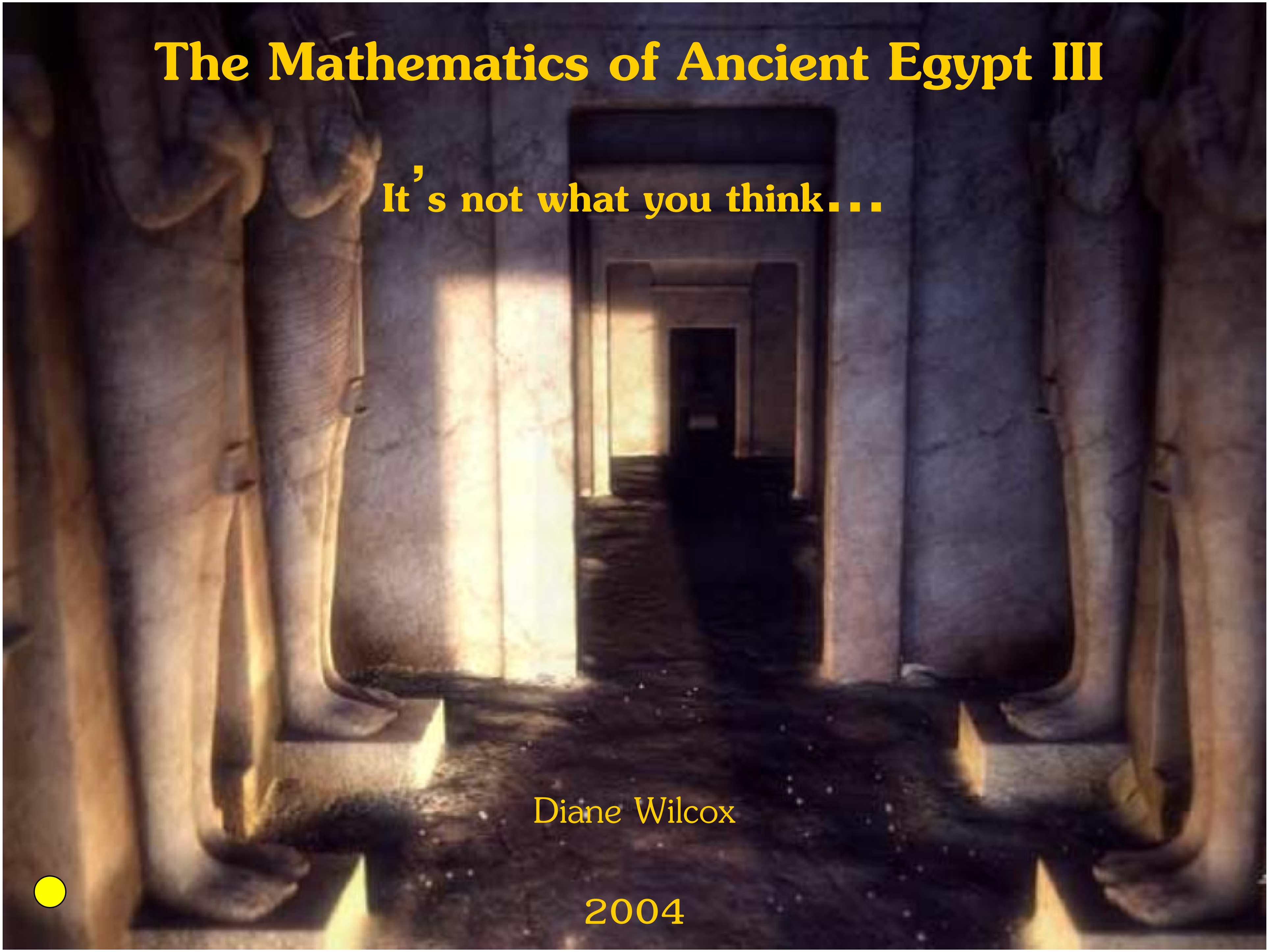


The Mathematics of Ancient Egypt III

It's not what you think...

Diane Wilcox

2004

How do computer game graphics work ?

We need to ask: What is a matrix ?

A matrix is a rectangular array of numbers

For example:

$$\begin{bmatrix} 13 & 22 & 4 \\ 1 & 5 & 15 \\ 12 & 11 & 7 \\ 9 & 21 & 29 \end{bmatrix}$$

AND ANOTHER MATRIX :

$$\begin{bmatrix} 12 & 119 & 10 & 213 & 4 & 56 & 77 \\ 433 & 233 & 11 & 45 & 289 & 7 & 0 \\ 0 & 45 & 56 & 99 & 100 & 543 & 34 \\ 12 & 3 & 1 & 576 & 3 & 45 & 77 \\ 898 & 17 & 21 & 568 & 2 & 0 & 587 \\ 756 & 749 & 875 & 2 & 87 & 23 & 88 \end{bmatrix}$$

Dimension:

4 row and 3 columns

→ 4x3 matrix

$$\begin{bmatrix} 13 & 22 & 4 \\ 1 & 5 & 15 \\ 12 & 11 & 7 \\ 9 & 21 & 29 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 119 & 10 & 213 & 4 & 56 & 77 \\ 433 & 233 & 11 & 45 & 289 & 7 & 0 \\ 0 & 45 & 56 & 99 & 100 & 543 & 34 \\ 12 & 3 & 1 & 576 & 3 & 45 & 77 \\ 898 & 17 & 21 & 568 & 2 & 0 & 587 \\ 756 & 749 & 875 & 2 & 87 & 23 & 88 \end{bmatrix}$$

6x7 MATRIX

We can add two matrices if they have the same dimension:

$$\begin{bmatrix} 3 & 5 & 0 \\ 1 & 7 & 2 \\ 8 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ 5 & 3 & 2 \\ 3 & 6 & 0 \end{bmatrix}$$

by adding the entries

$$\begin{bmatrix} 3 & 5 & 0 \\ 1 & 7 & 2 \\ 8 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ 5 & 3 & 2 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 5+1 & 0+5 \\ 1+5 & 7+3 & 2+2 \\ 8+3 & 4+6 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 & 5 \\ 6 & 10 & 4 \\ 11 & 10 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 2 & 8 \\ 0 & 7 & 19 \\ 11 & 9 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ 5 & 3 & 2 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 3 & 13 \\ 5 & 10 & 21 \\ 14 & 14 & 2 \end{bmatrix}$$

Notation – extending our symbolic language

Matrix: A

Matrix entry: a_{ij}

row i

column j

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_{32} & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

This is the entry in row 3 and column 2

Notation – extending our symbolic language

Matrix: $A = (a_{ij})$

Addition: $C = A + B$

$= (a_{ij}) + (b_{ij})$

$= (a_{ij} + b_{ij})$

$= (c_{ij})$

What do matrices have to do with Computer games graphics ?

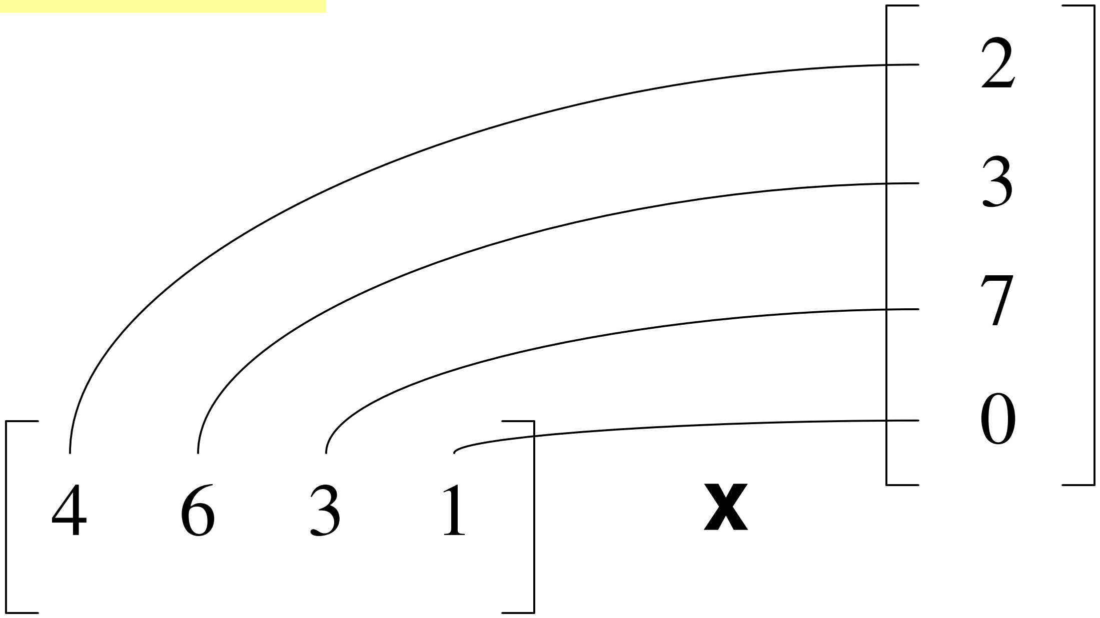
First we need to be able to multiply matrices:

The trick is to understand the definition of

the product of a row and a column

$$\begin{bmatrix} 4 & 6 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 7 \\ 0 \end{bmatrix}$$

the product of
a row matrix & a column matrix

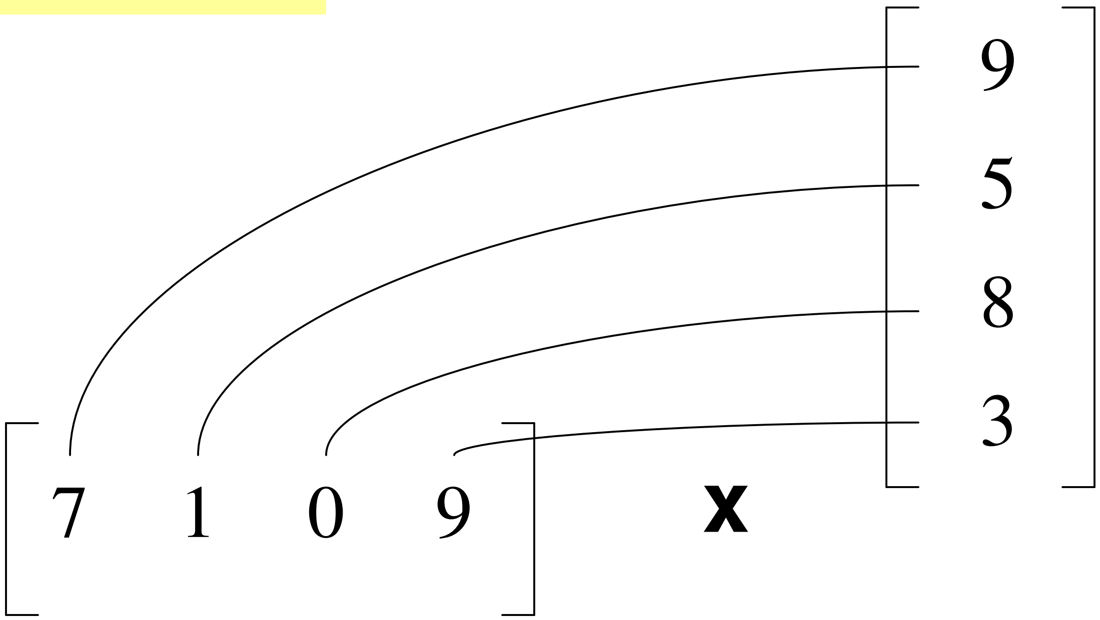


$$\begin{bmatrix} 4 & 6 & 3 & 1 \end{bmatrix} \quad \mathbf{x} \quad \begin{bmatrix} 2 \\ 3 \\ 7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{4x2} + \mathbf{6x3} + \mathbf{3x7} + \mathbf{1x0} \end{bmatrix}$$

$$= \begin{bmatrix} 47 \end{bmatrix}$$ = a number !

the product of a row matrix & a column matrix



$$\begin{bmatrix} 7 & 1 & 0 & 9 \end{bmatrix} \quad \mathbf{x} \quad \begin{bmatrix} 9 \\ 5 \\ 8 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{7\text{x}9 + 1\text{x}5 + 0\text{x}8 + 9\text{x}3} \end{bmatrix}$$

$$= \begin{bmatrix} 95 \end{bmatrix}$$

Product of 2x2 square matrix with 2x1 column matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = ???$$

Product of 2x2 square matrix with 2x1 column matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

1st row →

$$\begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \end{bmatrix}$$

2nd row →

$$\begin{bmatrix} c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx+dy \end{bmatrix}$$

Product of 2x2 square matrix with 2x1 column matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} = ???$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

1st row **X** *column*
a number !

2nd row **X** *column*
a number !

Product of 2x2 square matrix with 2x1 column matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ **transforms** $$\begin{bmatrix} x \\ y \end{bmatrix}$$

into $$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

Which is also a
2x1 column matrix

Example #1 using $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1.x+0.y \\ 0.x+1.y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}$$

Example #2 using

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1.x+0.y \\ 0.x+0.y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 0 \end{bmatrix}$$

About Example #2

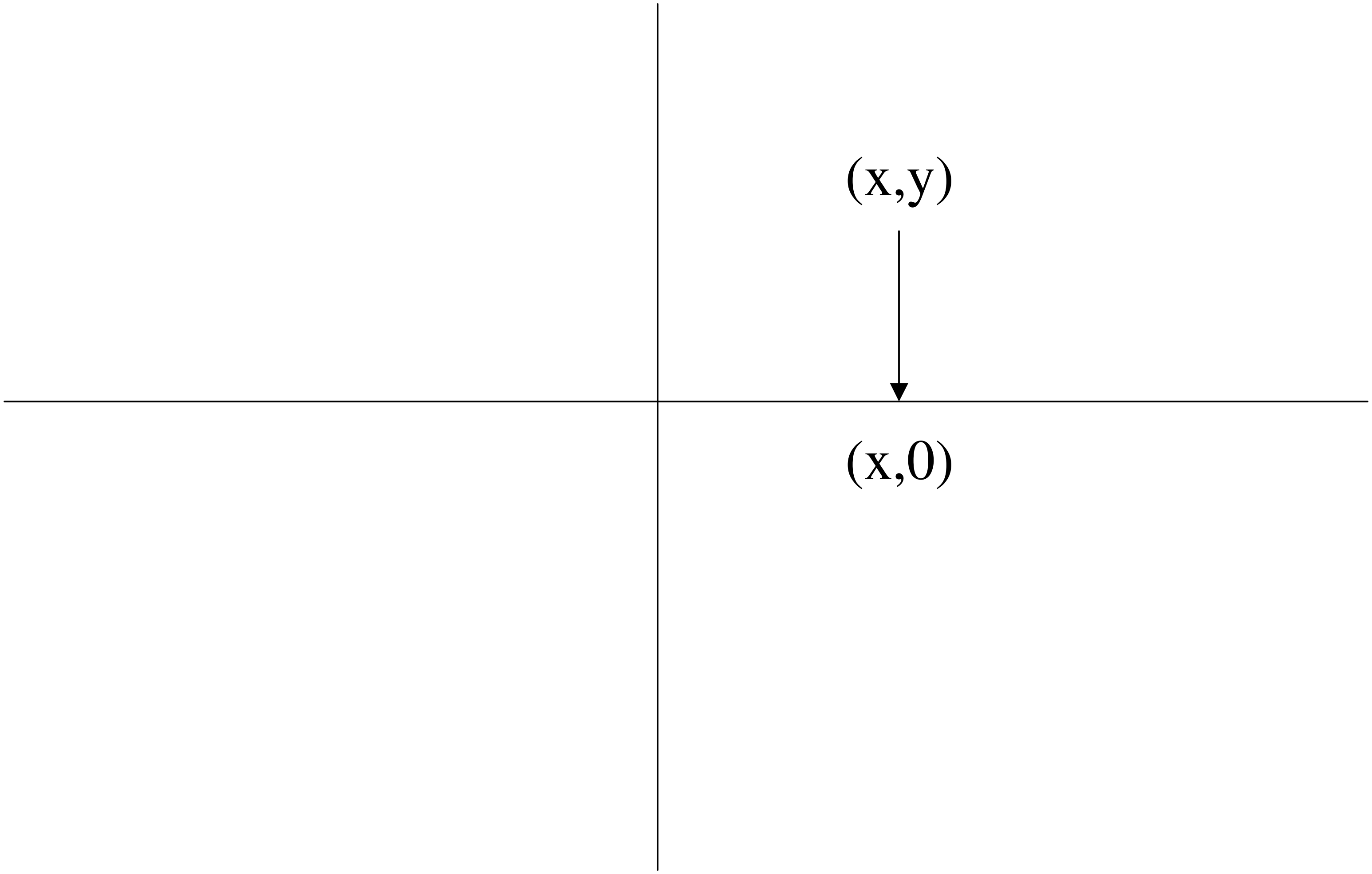
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \textbf{ transforms } \begin{bmatrix} x \\ y \end{bmatrix} \textbf{ into } \begin{bmatrix} x \\ 0 \end{bmatrix}$$

We can also think of this

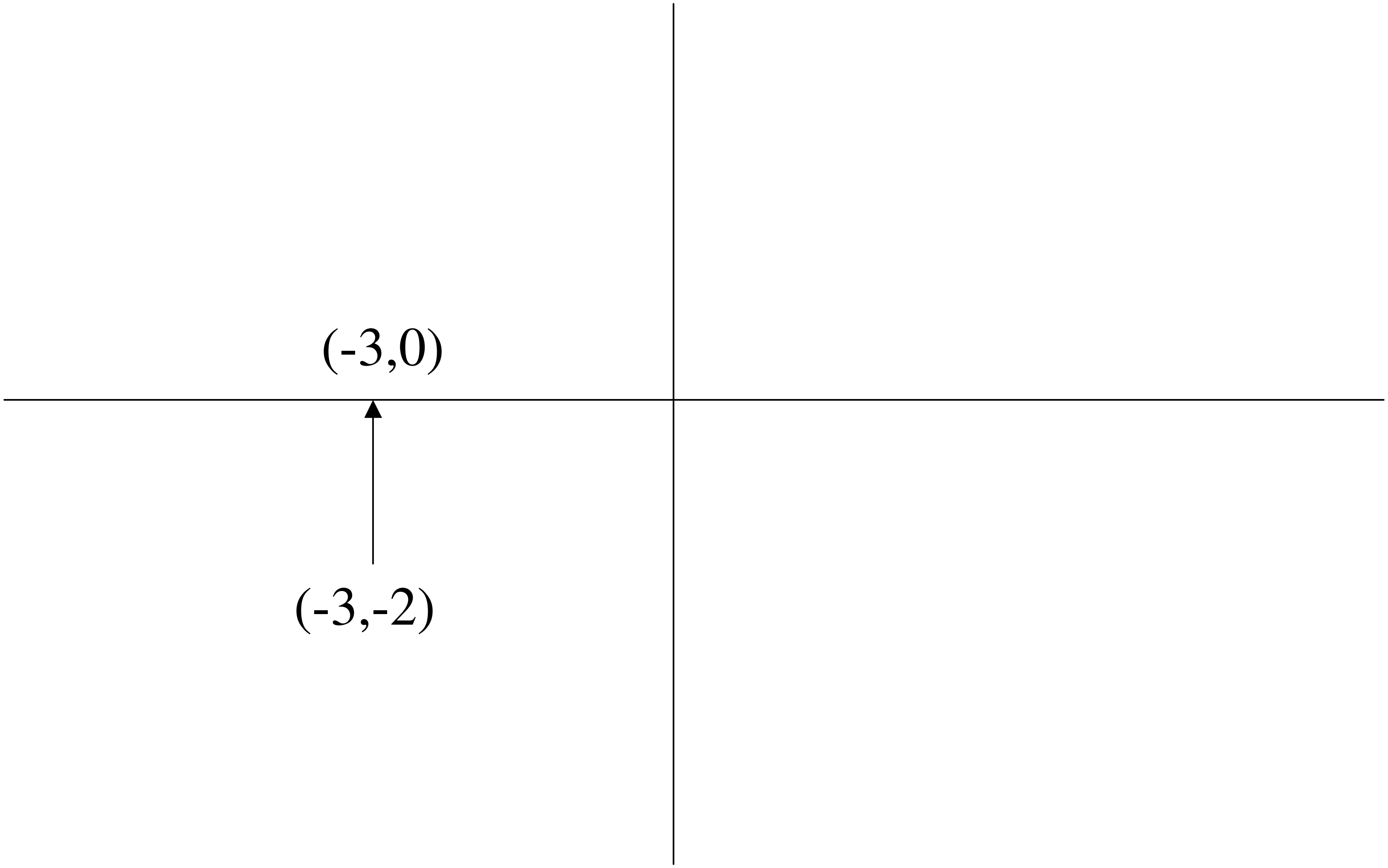
as a way of transforming

(x,y) into (x,0)

Graphical view of Example #2: Projection onto X-axis



PROJECTION ONTO X-AXIS : (x,y) IS TRANSFORMED INTO (x,0)



Example #3 using $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0.x+0.y \\ 0.x+1.y \end{bmatrix}$$

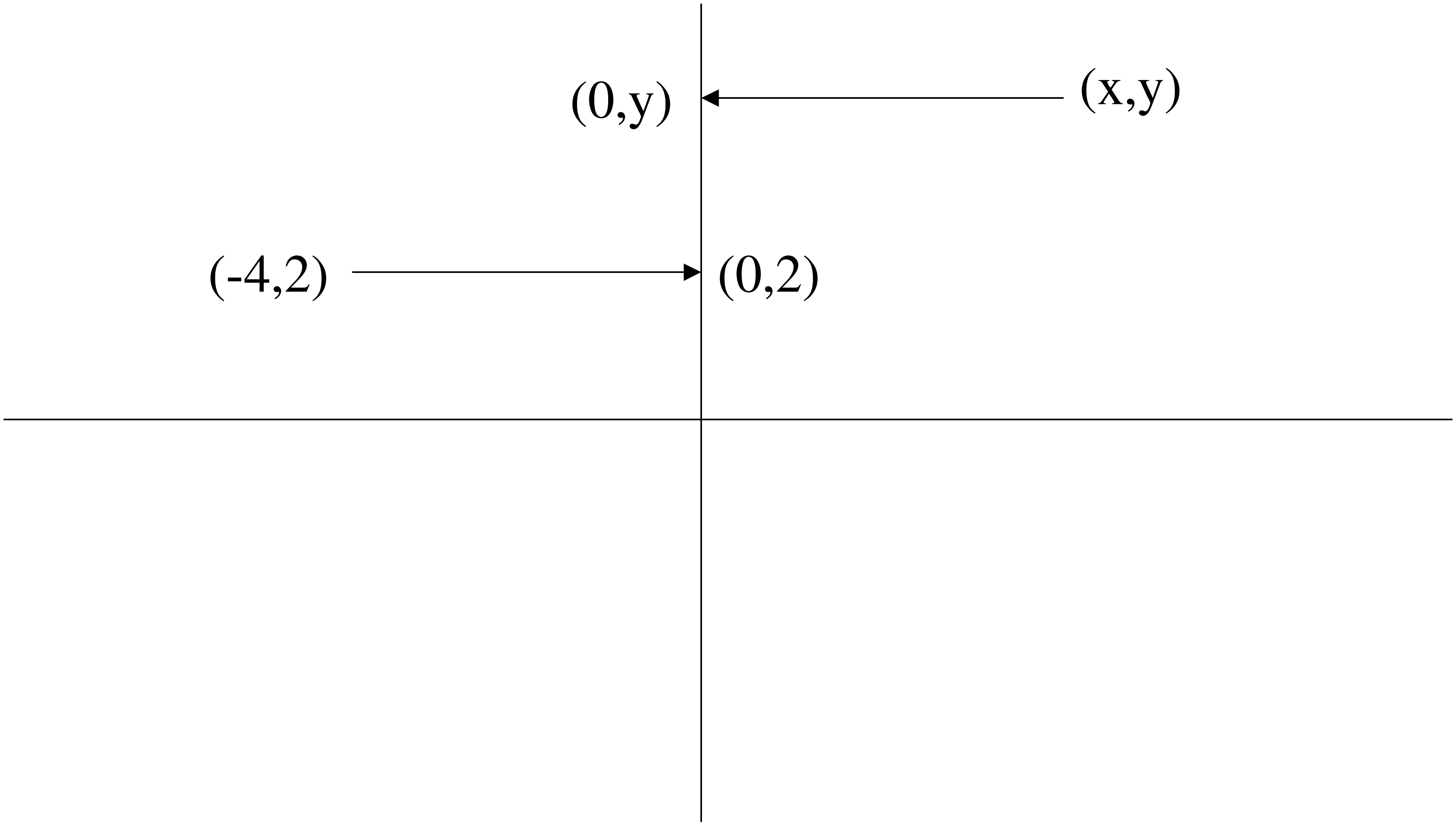
$$= \begin{bmatrix} 0 \\ y \end{bmatrix}$$

About Example #3

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \textbf{ transforms } \begin{bmatrix} x \\ y \end{bmatrix} \textbf{ into } \begin{bmatrix} 0 \\ y \end{bmatrix}$$

HERE (x,y) IS TRANSFORMED INTO (0,y)

PROJECTION ONTO Y-AXIS : (x,y) IS TRANSFORMED INTO (o,y)



Example #4 using $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2.x+0.y \\ 0.x+1.y \end{bmatrix}$$

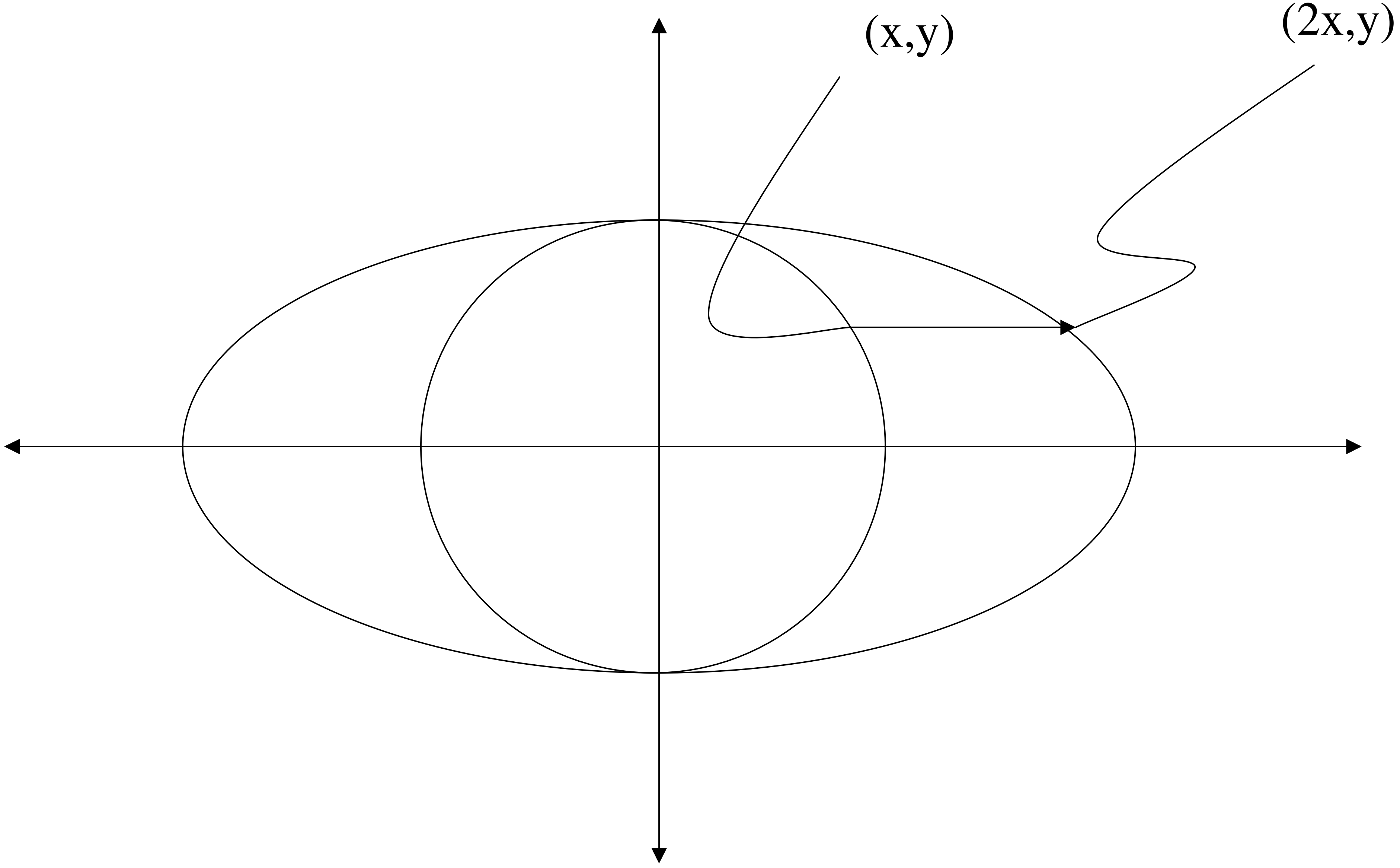
$$= \begin{bmatrix} 2x \\ y \end{bmatrix}$$

About Example #4

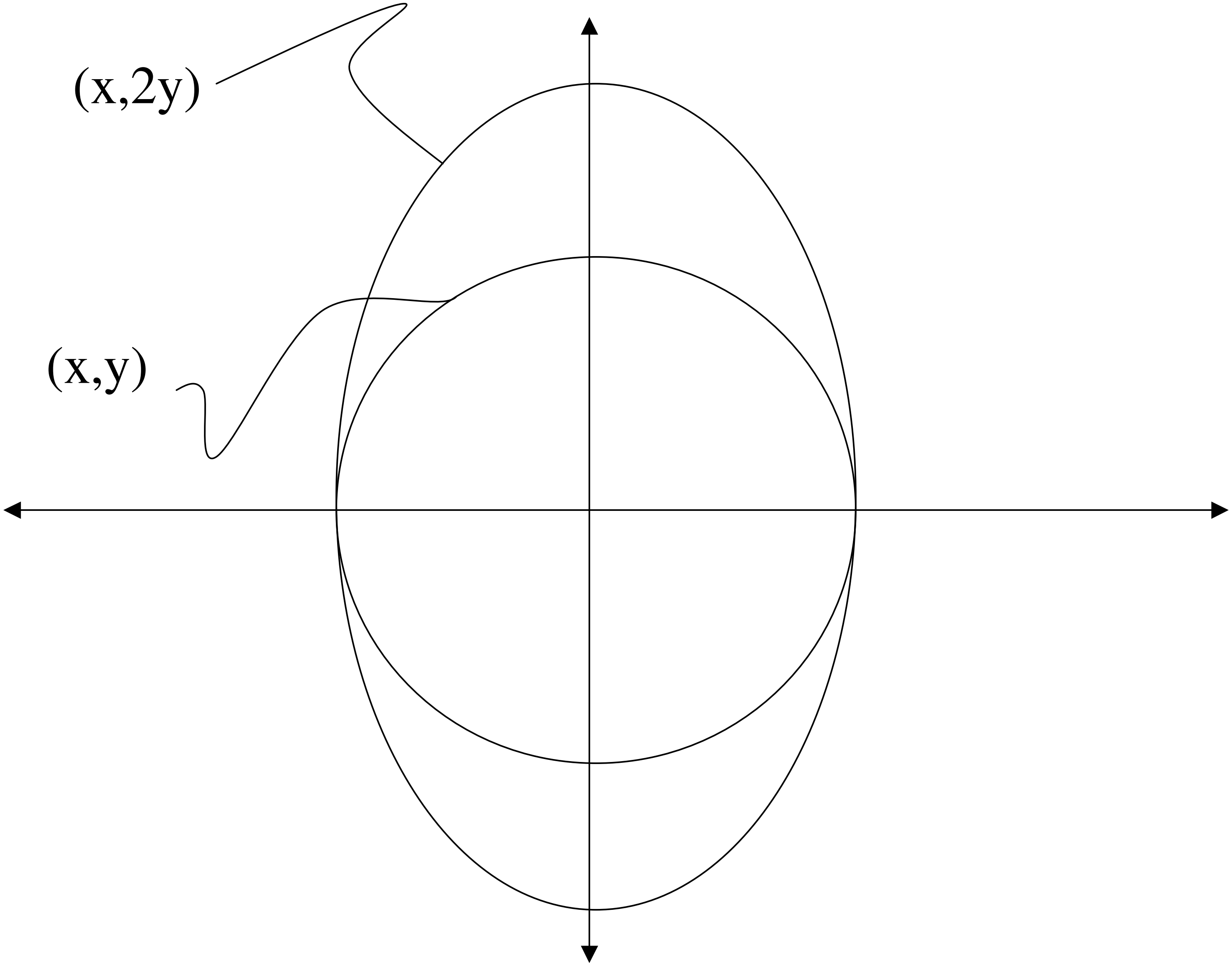
$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \textbf{ transforms } \begin{bmatrix} x \\ y \end{bmatrix} \textbf{ into } \begin{bmatrix} 2x \\ y \end{bmatrix}$$

Here (x,y) is transformed into (2x,y)

Graphical view of Example #4 : Horizontal stretching



Vertical stretching: which matrix would do this?



Example #5 using

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1.x+0.y \\ 3.x+1.y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 3x+y \end{bmatrix}$$

About Example #5

$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ **transforms** $\begin{bmatrix} x \\ y \end{bmatrix}$ **into** $\begin{bmatrix} x \\ 3x+y \end{bmatrix}$

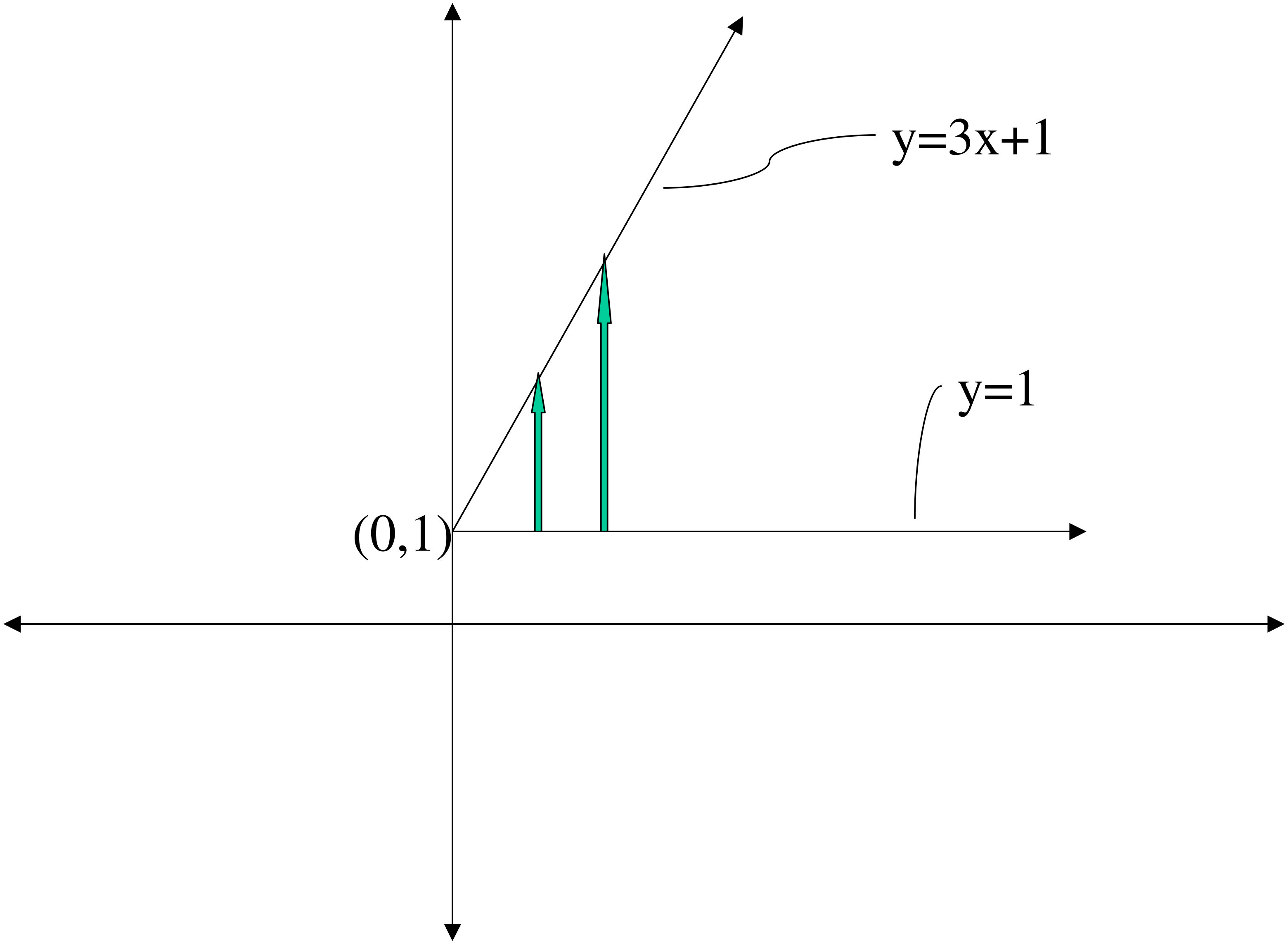
Here (x,y) is transformed into (x, 3x+y)

INTERPRETING THE TRANSFORMATION OF (x,y) INTO (x, 3x+y)

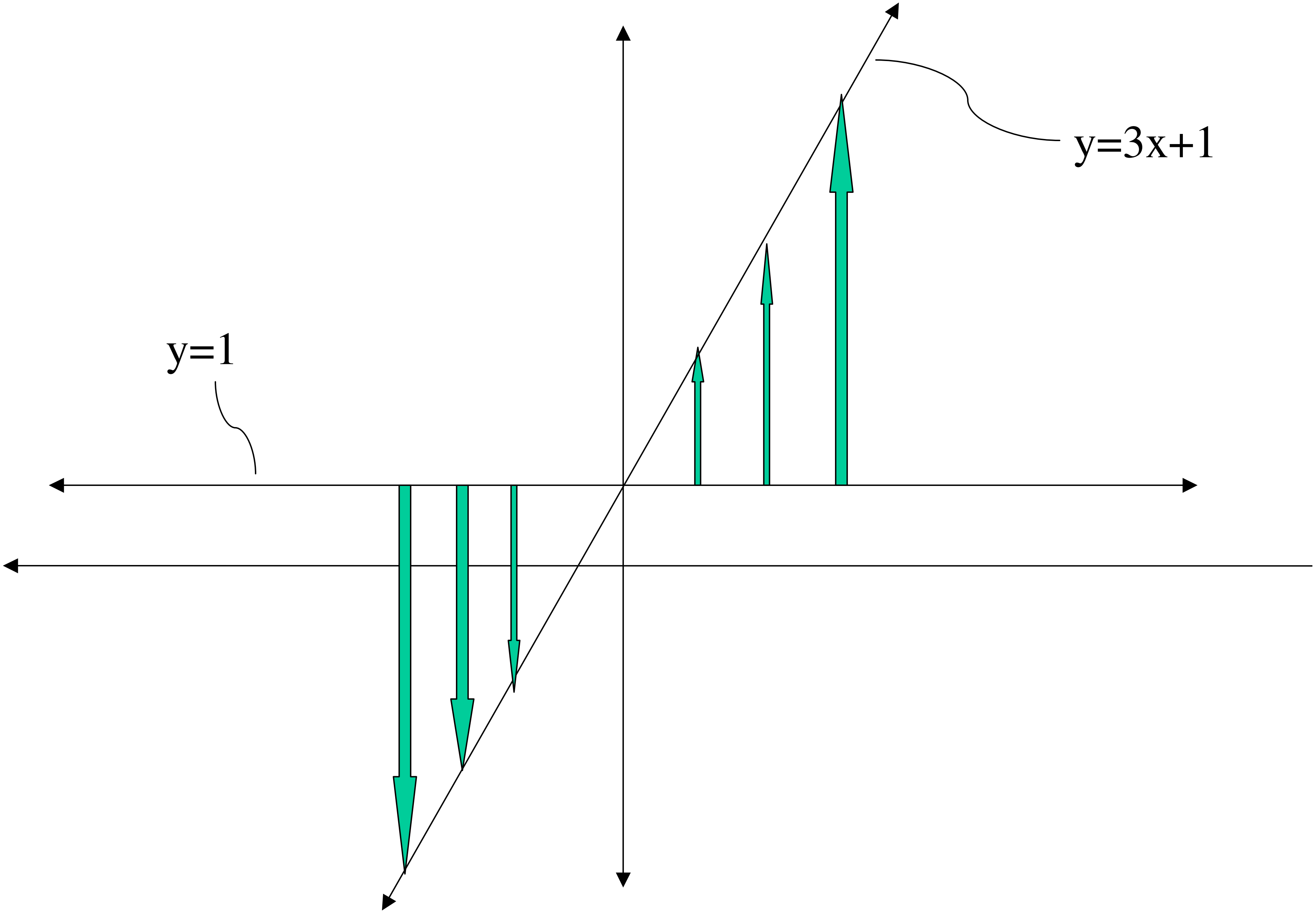
- Suppose y = 1 .
- Then (x,1) is transformed into (x, 3x+1).
- So our **y-coordinate** is **transformed**

from **y = 1** to **y = 3x + 1**

Graphical view of Example #5 : Shearing



Graphical view of Example #5 : Shearing



Example #6 using

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

$$\begin{bmatrix} \cos(\mathbf{45^o}) & -\sin(\mathbf{45^o}) \\ \sin(\mathbf{45^o}) & \cos(\mathbf{45^o}) \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} (1/\sqrt{2})\mathbf{x} - (1/\sqrt{2})\mathbf{y} \\ (1/\sqrt{2})\mathbf{x} + (1/\sqrt{2})\mathbf{y} \end{bmatrix}$$

About Example #6

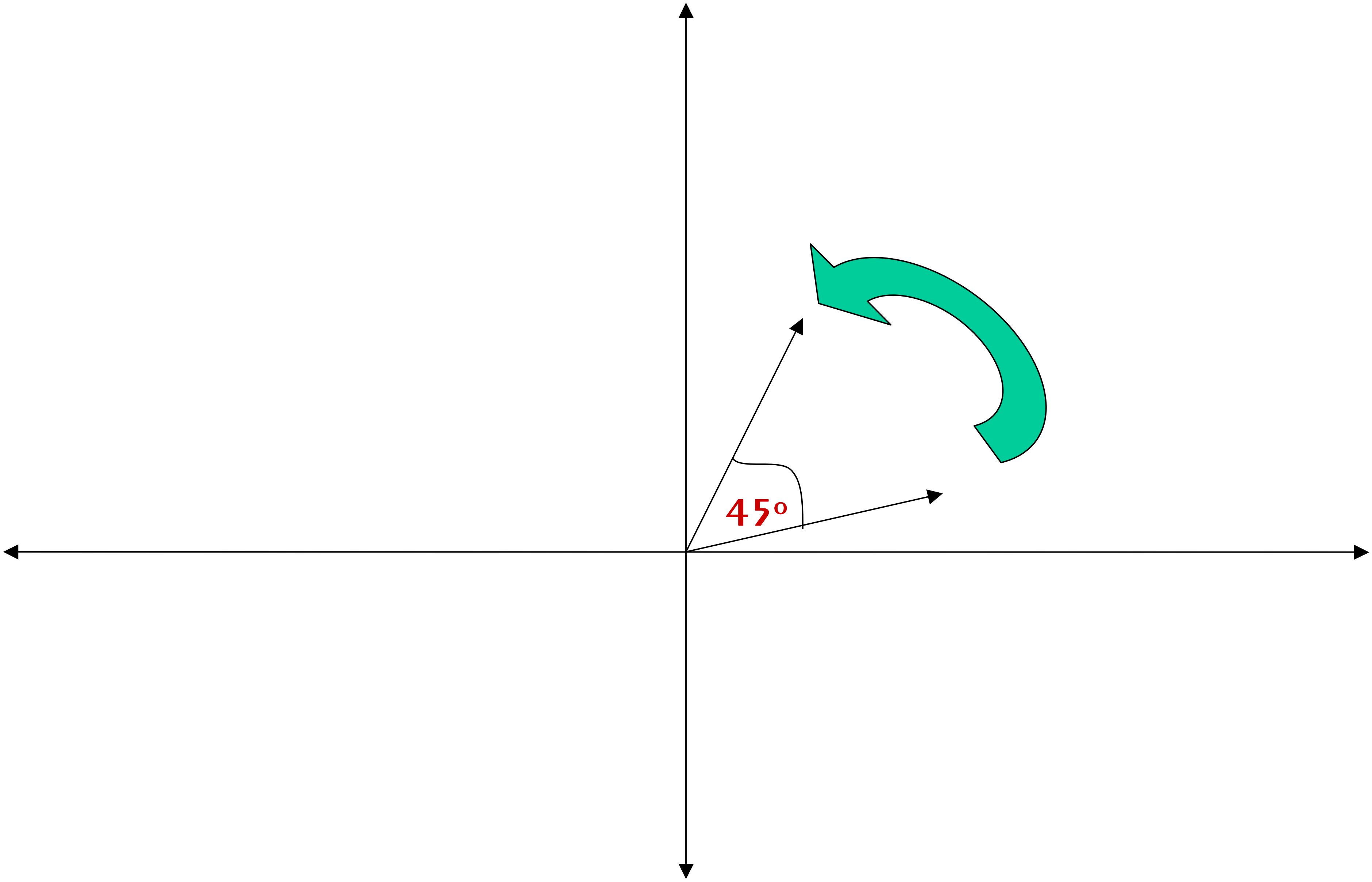
$$\begin{bmatrix} \cos(\mathbf{45^o}) & -\sin(\mathbf{45^o}) \\ \sin(\mathbf{45^o}) & \cos(\mathbf{45^o}) \end{bmatrix} \quad \textbf{transforms} \quad \begin{bmatrix} x \\ y \end{bmatrix} \quad \textbf{into}$$

New x-coordinate

$$\begin{bmatrix} (1/\sqrt{2}).\mathbf{x} - (1/\sqrt{2}).\mathbf{y} \\ (1/\sqrt{2}).\mathbf{x} + (1/\sqrt{2}).\mathbf{y} \end{bmatrix}$$

New y-coordinate

Graphical view of Example #6 : Rotation by 45°



About Example #6

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \textbf{ transforms } \begin{bmatrix} x \\ y \end{bmatrix} \textbf{ into}$$

$$\begin{array}{cl} \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix} & \begin{array}{l} \leftarrow \textit{New x-coordinate} \\ \leftarrow \textit{New y-coordinate} \end{array} \end{array}$$

ROTATION by θ: WORK OUT THE MATRIX FOR YOUR OWN CHOICE OF θ